Stability Properties of Nonlinear Delay Systems and the Breakdown of the Adiabatic Approximation

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We consider the adiabatic approximation as it is applied to a familiar class of infinite-dimensional equations. We show how this approximation may be directly invalidated by analyzing a statistical form of the eigenvalue equation before and after the adiabatic approximation is made. We introduce a statistical quantity which can be used to predict the existence of periodic windows in the chaotic regime.

Key words: Stability, Chaos, Adiabatic approximation.

1. Introduction

Adiabatic elimination of dynamical variables plays a central rôle in the study of laser instabilities and, more generally, in the field of nonlinear dynamics. Here we consider this standard technique when it is applied to a familiar class of delay-differential equations (DDE). This class of equations (see Eq. (1) below) are used to model such systems as semiconductor lasers coupled to external cavity systems [1–3]. These equations are infinite-dimensional due to the delay term. Understanding and characterizing the stability properties of these equations has proved difficult due to the lack of rigorous mathematics for this class. Consequently, the characterization of nonlinear delay systems has depended heavily on computational results.

In this paper we compute the real and imaginary parts of the eigenvalues of the linearized form of (1) before and after an adiabatic approximation is made. We show, by statistical methods, how our analysis may directly and simply invalidate the taking of this approximation (or limit) for this class of equations. This statistical approach may also be very useful in the determination of periodic regimes where mostly chaotic solutions are predicted.

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2. Delay Equations

Formally, we consider the following class of well-studied infinite-dimensional nonlinear equations:

$$\gamma^{-1}\,\mathrm{d}x(x)/\mathrm{d}t = f(x(t-t_r);\mu) - \eta\, x(t), \qquad (1)$$

where μ is the bifurcation parameter; γ (the Debye relaxation rate) and η are constants. The first term on the right hand side of (1) represents (nonlinear) feedback with delay time t_r.

In this paper we consider three specific functional forms for $f(x;\mu)$. Firstly, consider the physical system depicted in Fig. 1, where coherent light is incident on a ring system. A cell which contains a two level absorptive medium (e.g. a gas) is inserted in one of the arms of the cavity. If we assume that the atomic system is homogeneously broadened and that the transition frequency of the atoms coincide with that of the incident field, then in the dispersive limit and under strong dissipation $f(x;\mu)$ may be written [4, 5]:

$$f(x) = \pi\mu[1-\xi\cos(x+x_B)], \qquad (2)$$

where μ is proportional to the power of the incident light, and ξ represents the dissipation of the electromagnetic field in the cavity. In this case the variable x in (1) represents the phase shift suffered by the electric field in the medium and x_B the linear phase shift across the medium. The delay t_r is the time required for light to make a roundtrip in the cavity.

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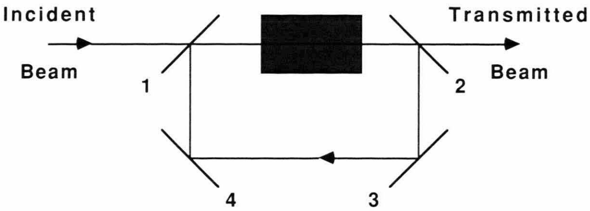

Fig. 1. A typical ring cavity system containing a two level medium. Mirrors 1 and 2 are partially transparent.

Table 1. Parameter values used for the systems considered.

Eq. (2)	$\mu = 1.01$	$\eta = 1.0$	$\xi = 0.96$	$x_b = -\frac{\pi}{2}$
Eq. (3)	$\mu = 0.2$	$\eta = 0.1$	$\varepsilon = 10.0$	
Eq. (4)	$\mu = 4.0$	$\eta = 1.0$		

For the physiologically-inspired Mackey-Glass equation $f(x; \mu)$ in (1) takes the form [6]:

$$f(x; \mu) = \mu\, x(t-t_r)/(1+[x(t)]^{\varepsilon}), \tag{3}$$

where ε is a constant. In our study we shall also use $f(x; \mu)$ given by

$$f(x; \mu) = \mu\, x(t-t_r)(1-x(t-t_r)). \tag{4}$$

which is a continuous form of the well-studied logistic map. Table 1 contains the parameter values used for the three different cases of $f(x; \mu)$ presented here. These values were chosen such that (1) predicted chaotic behaviour for nearly all t_r, as in this paper we are only concerned with the effect of the delay time.

3. The Adiabatic Approximation

The adiabatic procedure [4, 8] entails the explicit elimination of rapidly adjusting variables. Elimination of rapidly relaxing variables in a system's dynamics can lead to a considerable reduction in complexity of a problem, sometimes to the point that analytic solutions become accessible. In a semiconductor laser, for example, we assume that induced polarization relaxes much faster than any of the other variables (e.g., the cavity electromagnetic field or the population inversion). Consequently, we take the stationary value of polarization, so reducing the number of coupled differential equations.

If we rescale time by letting $\tau = t/t_r$, (1) may be written:

$$\mathrm{d}x(\tau)/\mathrm{d}\tau = \alpha[f(x(\tau-1); \mu; \eta) - x(\tau)], \tag{5}$$

where $\alpha = \gamma t_r \eta$. In the limit $\alpha \to \infty$ [which for example corresponds to the medium responding to the electric field adiabatically (or instantaneously) in the system of Fig. 1] we get from (5) either

$$\mathrm{d}x(\tau)/\mathrm{d}\tau \to \infty \quad \text{or} \quad [f(x(\tau-1)) - x(\tau)] \to 0\,.$$

A one-dimensional iterative map viz. $x(\tau) = f(x(\tau-1))$ is derived from the latter limit and constitutes an *adiabatic approximation.* This approximation, which leaves the differential undefined, has been assumed valid in some studies (e.g. [8]). However, the validity of this approximation has been brought into question by the work of Le Berre et al. [9]. We also question the validity of this approximation by considering a very different approach than that made in [9]. Our approach compares statistical forms of the linearized equation before and after the adiabatic approximation is made.

4. Linear Stability Analysis

In the following we make the common assumption that dynamical states separated by a roundtrip time have identical stability properties. A linear stability analysis of (5) yields the characteristic equation:

$$\lambda = \alpha\{f'(x_\tau)\exp(-\lambda) - 1\}, \tag{6}$$

where the notation $f'(x_\tau) = \mathrm{d}f(x_\tau)/\mathrm{d}x_\tau$ and $x_\tau = x(\tau-1)$ is used. For the map-model which may be written as

$$x_{n+1} = f(x_n; \mu; \eta), \tag{7}$$

where $n\,(=n\tau)$ now plays the rôle of the time variable, a linear stability analysis yields the characteristic equation:

$$1 = f'(x_n)\exp(-\lambda). \tag{8}$$

Letting $\lambda = a + i\omega$ in (6) gives:

$$a = \alpha\{f'(x_\tau)\exp(-a)\cos\omega - 1\}, \tag{9}$$

$$\omega = -\alpha f'(x_\tau)\exp(-a)\sin\omega. \tag{10}$$

Equations (9) and (10) have an infinite number of discrete solutions.

By letting $\lambda = a + i\omega$ in (8) we find:

$$a = \log|f'(x_n)|, \quad f'(x_n) \neq 0, \tag{11}$$

$$\omega = n\pi, \quad n \in \mathbb{Z}. \tag{12}$$

Note that (9) and (10) reduce (smoothly) to (11) and (12) in the singular limit $\alpha \to \infty$. It is also easy to show that a is bounded by $\log|f'(x_\tau)|$ in (9), so that for relatively large α the condition $\alpha \gg a$ is well fulfilled (since $\log|f'(x_\tau)| \sim 1$ for our equations). However, while the allowed values of ω collapse on to $\delta(\omega - n\pi)$ at the adiabatic limit, ω is not bounded and therefore its effect at large α may not be ignored. We may only say, from (10), that (ω/α) is bounded. In the next section we shall be concerned with Lyapunov characteristic exponents (LCE) which are defined for most if not all dynamical systems. As a working definition we shall say that a system is chaotic (in the sense that is very sensitive to initial conditions) if it has at least one positive Lyapunov exponent.

5. Lyapunov Characteristic Exponent

From (11) the Lyapunov characteristic exponent [10] for the map-model is defined as

$$\chi = \langle a \rangle = \langle \log|f'(x_n)| \rangle , \qquad (13)$$

where the angled brackets refer to the time average with respect to the sequence $\{x_n\}$. Formally, if $\chi > 0$ the sequence $\{x_n\}$ is said to be chaotic. By performing a similar averaging procedure we write, in analogy with (13), (9), and (10) for the (continuous) infinite-dimensional case as

$$\langle a \rangle_{\text{cont}} + \langle (1/2)\log(1+\beta) \rangle = \mathscr{L} , \qquad (14)$$

where the subscript "cont" distinguishes this case from the discrete one, $\beta = 2(a/\alpha) + (a/\alpha)^2 + (\omega/\alpha)^2$ and $\mathscr{L} = \langle \log|f'(x_\tau)| \rangle$. Note the similarity between the definition of χ and $\mathscr{L}$. We shall show later that the statistical quantity $\mathscr{L}$ is very useful as it allows us to predict regions of periodic motion within chaos. Note also that $\mathscr{L}$ is computed from (1) before any approximation is taken, whereas χ is calculated from the mapping i.e. after the adiabatic approximation is made. Below we shall compare the numerical values computed for these two averaged quantities.

Table 2. The LCEs for the map-models and $\mathscr{L}$ for the continuous functions computed using the values given in Table 1.

α	10^3	5×10^3	10^4	∞ (i.e. χ) (nats/iteration)*
Eq. (2)	0.35 ...	0.35 ...	0.35 ...	0.56 ...
Eq. (3)	0.63 ...	0.63 ...	0.63 ...	0.51 ...
Eq. (4)	0.40 ...	0.40 ...	0.40 ...	0.69 ...

* This unit is equivalent to nats/roundtrip or nats/(unit time).

In (14) as α becomes large $\beta \approx (\omega/\alpha)^2$ (since a is bounded) so that we cannot compute $\langle a \rangle_{\text{cont}}$ directly. Furthermore, it is easy to show that the condition $\langle a \rangle_{\text{cont}} < \mathscr{L}$ must hold. In the next section we shall compute $\mathscr{L}$ numerically and show it to be invariant to changes in α (once $\alpha \gg 1$) thus providing an upper limit for $\langle a \rangle_{\text{cont}}$. This allows us to compare χ and $\mathscr{L}$ directly.

6. Computational Results

In general, the averaged quantity defined by (13) converges to a well-defined value independent of initial conditions. Table 2 summarizes (in the right hand column) the values of χ computed for each of the (adiabatically-approximated) map-models obtained for the three functions defined above.

When computing an averaged quantity, such as $\mathscr{L}$ in (14), it is impossible to follow trajectories on a chaotic attractor for very long times with any accuracy. It is, on the other hand, possible to compute statistical averages. By varying both the number of roundtrips allowed for transients and the number of roundtrip used, $\mathscr{L}$ was discovered to converge to a well-defined value independent of initial conditions (for fixed values of the parameters). The number of transient roundtrip typically allowed was 5×10^3 (using initial constant functions across the interval) with typical numbers of useful roundtrips $\sim 10^3$. It should be emphasized that all quantities comprising the numerical algorithm were varied to assure a proper statistical convergence of $\mathscr{L}$.

All integrations were performed with a Runge-Kutta algorithm where the stepsize ($\Delta\tau$) was kept as small as possible. However as can be seen from (5), the quantity of computational interest is $\alpha\Delta\tau$ which was maintained $\gtrsim 10^{-1}$ in order to insure integrating inside the Debye relaxation time (and to keep the global error as low as possible). For $\alpha = 10^4$ this means $\Delta\tau \gtrsim 10^{-5}$.

Table 2 also summarizes the values obtained for $\mathscr{L}$ for the functions defined by (2), (3), and (4). In all cases the value of $\mathscr{L}$ remained the same for the values of α chosen. We believe that this invariance holds for all finite α (once $\alpha \gg 1$). Since the quantity χ is computed from the sequence $x(\tau), x(\tau+1), \ldots$, we similarly computed $\mathscr{L}$ from this sequence with no change in the final result. These results indicate that, particularly in the cases of (2) and (4), the stability properties of the DDEs are undoubtedly different from those of the map since $\langle a \rangle_{\text{cont}}$ in these cases could not continuously approach $\langle a \rangle$ (i.e. χ). (We do not expect the Mackey-Glass equation to be different.) Thus the stability properties of the continuous and discrete cases must remain different until the adiabatic limit is imposed and these separate cases never approach one another except (discontinuously) at this limit. This indicates, in agreement with the conjecture of Le Berre et al. [9], that the adiabatic limit is invalid.

7. Periodic Windows

Next we consider the value of $\mathscr{L}$ when periodic behaviour is predicted by (1). Figure 2 shows $\mathscr{L}$ computed in the range $15 \leq \alpha \leq 20$ using the function defined by (4). A well-defined dip is observed around $\alpha = 18$. This dip was shown to correspond to a periodic window. From Fig. 2 the chaotic region has a $\mathscr{L}$ that is approximately 0.40 while in the periodic region it has a value of about 0.32. All windows above this parameter range exhibited the same behaviour in that chaotic behaviour produced the same $\mathscr{L}$ (to within a few percent) while periodic behaviour predicted a lower, but also well-defined, $\mathscr{L}$. It is very easy to see by observing $\mathscr{L}$ (once α is not too small) whether the system is undergoing chaotic, periodic or transient behaviour. Identical results were obtained for the functions defined by (2) and (3) where in all cases $\mathscr{L}$ (periodic) was appreciably less than $\mathscr{L}$ (chaotic).

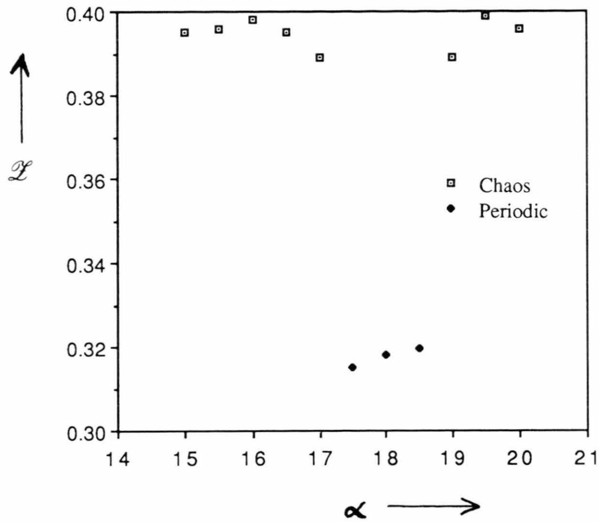

Fig. 2. $\mathscr{L}$ computed for the continuous logistic-type equation. The solid diamonds correspond to a periodic window.

8. Discussion

The Lyapunov characteristic exponents of delay differential equations can be computed using the algorithm outlined by Farmer [6]. In addition, there is some evidence that the metric entropy and information dimension may be expressed in terms of the spectrum of Lyapunov exponents. Kaplan and Yorke [11, 12] conjectured that the Lyapunov dimension is equal to the information dimension (D_I). From this conjecture a continuous flow system that is chaotic must always have a dimension greater than two. In contrast, the attractor of a one-dimensional (map) system that is also chaotic has a dimension of less than one. Assuming the validity of the Kaplan and Yorke conjecture, an immediate discrepancy in dimensionality is predicted between the continuous and discrete cases without the need for any numerical support. Therefore, this conjecture cannot be used to test the validity of the adiabatic approximation outlined in Sect. 3 above.

Nevertheless, using the Kaplan and Yorke conjecture the dimension of the chaotic attractor is found to increase linearly with α for the class defined by (1) [6, 9, 13]. In particular, for the system defined by (2), we have determined that in the chaotic region D_I follows the relation:

$$D_I = 0.61\,\alpha + 0.63, \tag{15}$$

for $6 \leq \alpha \leq 30$ (using the parameter values given in Table 1). Going to higher α proves extremely expensive computationally. In the limit $\alpha \to \infty$, D_I is therefore assumed to go to infinity. Le Berre et al. use this linear extrapolation, in part, to predict that the adiabatic approximation is not valid for delay-differential equations.

In this paper we have attempted to observe a direct transition to the adiabatic limit using (14). For this purpose we have introduced the quantity $\mathscr{L}$ in analogy with the Lyapunov characteristic exponent of the one dimensional system. This exponent is just the time averaged real part of the eigenvalue of the linearized equation. For delay systems the LCEs are more difficult to compute and direct comparison with the one-dimensional system cannot be made. Therefore the one-to-one correspondence created by (14) is of more value, especially as analytical continuity is retained.

We noted in (14) that as α is increased the condition $\alpha \gg a$ is well specified. However once α remains finite, ω is never decoupled from α and correspondingly plays an important and significant rôle in the stability properties of the solutions. According to our numerical results $\langle a \rangle_{\text{cont}}$ can never approach χ except at the limit, which therefore makes this limit a singular discontinuity. This shows that the stability properties of the chaotic solutions for the DDE and the map remain markedly different indicating that the differential term may never be left unspecified in these delay systems. In the system schematically represented in Fig. 1, $\alpha = 10^4$ corresponds to the roundtrip in the external cavity (t_r) being 10^4 times larger than the response time of the two level medium ($\tau_s = \gamma^{-1}$). In lower dimensional systems such timescale disparties would undoubtedly justify the use of an adiabatic approximation. However, in real delay physical systems we have shown that the map-limit can never said to be fulfilled. Consequently, it may only be useful for determined certain properties of the delay-differential equation.

Finally, the difference between the $\mathscr{L}$ for the chaotic and periodic waveforms is quite evident, thereby providing a convenient method for determining regions of chaotic behaviour. This method is far quicker than plotting waveforms, phase portraits or computing the Lyapunov spectra, especially as chaotic transients may be long-lived in these periodic regions [13]. Another advantage is that this method is unquestionably extendible up to regions computationally inaccessible to the Lyapunov spectra algorithm.

9. Conclusion

We have considered the adiabatic approximation as it is applied to a class of delay-differential equations. We have shown, by comparison between a direct calculation of the Lyapunov exponent for the map-model and an analogous equation for the continuous form, that the stability properties of the two remain markedly different (even with increasing delay parameter) indicating that the adiabatic limit represents a discontinuous transition. We have also shown that this form of analysis allows an effective method for determining periodic regimes in the continuous systems. We note in passing that the constancy of $\mathscr{L}$ in the chaotic region is most likely related to the constancy of the metric entropy first shown for the Mackey-Glass equation by Farmer [6].

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[1] J. O'Gorman, B. J. Hawdon, J. Hegarty, and D. M. Heffernan, Electron Lett. **25,** 114 (1989).
[2] J. O'Gorman, B. J. Hawdon, J. Hegarty, and D. M. Heffernan, Technical digest of Sixth Rochester Conference on Coherence and Quantum Optics, June 1989.
[3] K. Otsuka, J. Opt. Soc. Amer. **B 2,** 168 (1985).
[4] K. Ikeda, Optical Instabilities (R. Boyd, M. G. Raymer, and L. M. Narducci, eds.), Cambridge University Press, Cambridge 1986.
[5] F. A. Hopf, D. L. Kaplan, M. H. Rose, L. D. Sanders, and M. W. Derstine, Phys. Rev. Lett. **57,** 1394 (1986).
[6] J. D. Farmer, Physica **4 D,** 366 (1982) and references therein.
[7] K. Ikeda, K. Kondo, and O. Akimoto, Phys. Rev. Lett. **49,** 1467 (1982).
[8] H. Haken, Optical Instabilities (R. Boyd, M. G. Raymer, and L. M. Narducci, eds.), Cambridge University Press, Cambridge 1986.
[9] M. Le Berre, E. Ressayre, A. Tallet, and H. Gibbs, Phys. Rev. Lett. **56,** 274 (1986).
[10] P. W. Milonni, M.-L. Shih, and J. R. Akerhalt, Chaos in Laser-Matter, Interactions, World Scientific, Singapore 1987.
[11] J. Kaplan and J. Yorke, Lecture Notes in Mathematics, No. 730, Springer-Verlag, Berlin 1979, pp. 204.
[12] D. Ruelle, The validity of this conjecture for infinite-dimensional systems, however, has been questioned, private communication.
[13] B. J. Hawdon, J. O'Gorman, and D. M. Heffernan, J. Physique **49,** C2-419 (1988).